Advances in Structural Mechanics
Volume 1 Issue 1 No. 2
Published Online July 10 2022

Research on nonlinear static analysis of RC beams by shooting method

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Abstract: Taking nonlinear characteristics and shear deformation of steel and concrete into consideration, virtual work principle based on FEM is used to derive control differential equations and boundary conditions for static force analysis of RC beam. The control differential equations are transferred to six first order ordinary differential equations for nonlinear differential boundary value problems. Nonlinear shooting method is adopted to study on statics behavior for acquiring displacement and bending moment response. Compared with Timoshenko and Euler-Bernoulli analysis, deployment analysis and parameters are tested to be correct. Additionally, shear effect and moment analysis tests results in moment modulation coefficients.

Keywords: RC beam, material nonlinearity, timoshenko, nonlinear shooting method, shear effect

1 Introduction

Reinforced Concrete (RC) beam [1-2] has been extensively used as a structural element in civil engineering. The bearing capacity, stiffness and stabilization problems of RC analysis and design take a generally attention. Nowadays, finite element method (FEM) [3-4] increases to be a main method to analyze structural mechanics of RC beams.

Considering nonlinear constitutive relation and slim-lined construction characteristics of steel and concrete, axial displacement of RC girder is often determined by plane-section assumption [1]. Axial displacement of particles in cross-section of RC beam performs linear distribution along beam height. For tracing stress state in any section of RC beam cross-section and reasonably describing material nonlinearity of mechanical behavior, Taucer *et al* [5] proposes to apply fiber cross section method (FCSM) to do numerical integration of cross-section. FCSM hypothesizes that strain uniformly distributes in each fiber region when cross section is discretized. Therefore, division finesse of fiber in cross section remarkably influences on RC beam's mechanical behavior prediction. After that, researches of RC beams and columns based on fiber cross section method have been being too numerous to enumerate [6-9]. Nowadays, a professional FEM analysis program OpenSees [10] is inserted with nonlinear finite element of beam based on FCSM. Although finite element mechanical analysis of RC girder structure is successfully obtained under FCSM, some problems also happens. For example, stress analysis is rough in fiber cross section under FCSM assumption [11], development of computer program is as well uneasy on simple design and analysis of RC beam structure.

Shooting method [12] is running by properly choosing and adjusting initial conditions to solve a series of initial problems, and making those values to approach given boundary conditions. It is likely to continuously adjust shooting condition to make it reach at booking target. It is a numerical method to solve boundary value problems of ordinary differential equations [13], and features on simplifying development procedure and being effectively solved. In recent years, shooting method is extensively applied in nonlinear buckling analysis of structure [14], nonlinear dynamic analysis [15], and nonlinear static analysis of pile foundations [16].

Based on virtual work principle static analysis, control differential equations and boundary conditions of Timoshenko [8] RC beam are established with first-order shear deformation in deep beam considered. Furthermore, some control differential equations are derived with direct numerical solution by adopting nonlinear shooting method. It becomes a new tool was produced to design and analyze RC girder. In this paper, shooting solution of deflection analysis in RC beam is validated reasonably. shear deformation is also researched in deep RC beam. And, moment modulation coefficients (*M*) are analyzed by nonlinear material property.

In this text, developed mathematical models and calculation methods have three characteristics. Firstly, fiber discretization in RC beams' cross section is unwanted in process of applying nonlinear shooting method. Secondly, application scope of RC beam analysis enlarges in deep beam. Thirdly, actualization of numerical producer is more highly simplified than FEM.

2 Control differential equation

2.1 Euler equation

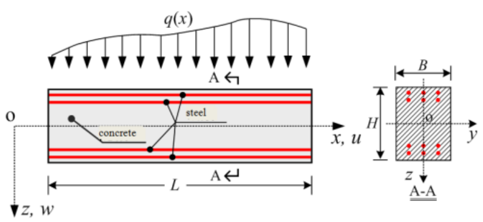


Fig. 1. Diagram of RC beam and its cross section

A RC beam is displayed in Fig.1 in Cartesian coordinates *oxyz* with length at *L*. The rectangular cross section shows height at H and width at B. There are some longitudinal steels paralleling with x-axis inside. The topside of beam bears a distributed load at $q(x)$ along positive z-direction in Cartesian coordinates xoz. The RC beam in equilibrium state has transverse displacement u and longitudinal displacement w. In order to simplify mathematical module, three assumptions are introduced as (I) assuming positive x-shift satisfies theory of Timoshenko beam, while relative slippage between steel and concrete is ignored; (II) there are only minor slope and deformation after RC beam is strained; (III) origin of Cartesian coordinates is set in centroid of cross section, while positive x-direction and z-direction are individually agreed with positive axial displacement and transverse displacement of particle in beam body. Before distorted, particle $u(x, z)$ has positive axial displacement as,

$$u(x, z)=u_0(x)-z\theta(x) \qquad (1)$$

Above $\theta(x)$ means clockwise rotation angle in cross section at x; $u_0(x)$ shows axial displacement of particle in centroid of cross section; $w(x)$ means deflection.

By applying small deformation geometric equation [17], axial positive strain $\varepsilon_c(x, z)$ and shearing strain $\gamma_c(x, z)$ of concrete are followed as,

$$\begin{cases} \varepsilon_{\mathrm{c}}(x,z)=\dfrac{\mathrm{d}u_0(x)}{\mathrm{d}x}-z\dfrac{\mathrm{d}\theta(x)}{\mathrm{d}x} \\ \gamma_{\mathrm{c}}(x,z)=\dfrac{\mathrm{d}w(x)}{\mathrm{d}x}-\theta(x) \end{cases} \qquad (2)$$

Considering that there is no relative slippage between concrete and steel, direct strain of steel at z_i is shown as,

$$\varepsilon_{si}=\varepsilon_{ci}=\varepsilon_{\mathrm{c}}(x,z_i) \qquad (3)$$

Without shear strain of steel, virtual work principle can demonstrate by equation (4) for RC beam static analysis [18].

$$\int_{x_0}^{x_1}\left\{\iint_A[\sigma_{\mathrm{c}}\delta(\varepsilon_{\mathrm{c}})+\kappa\tau_{\mathrm{c}}\delta(\gamma_{\mathrm{c}})]\mathrm{d}A + \sum_{i=1}^{N}A_i\sigma_{si}\delta\varepsilon_{si}\right\}\mathrm{d}x=\int_{x_0}^{x_1}q(x)\delta w\mathrm{d}x \qquad (4)$$

Above σ_{si}, A_i and ε_{si} mean direct strain, cross section area and positive deformation of steel i, respectively; N means number of steel; σ_c and ε_c are respectively equal to x-direct positive stress and deformation in steel; τ_c and γ_c separately display x-direct positive stress and deformation in concrete; δ is a variational operator; $[x_0, x_1]$ means x-Coordinate interval; κ is the shearing stress correction coefficient. When it comes to rectangular section, value of κ is 5/6。

For monotonic loading problems, stress-strain relationship of concrete or steel can be described by nonlinear functions f_1, f_2 and f_3 as,

$$\sigma_{\mathrm{c}}=f_1(\varepsilon_{\mathrm{c}}),\ \tau_{\mathrm{c}}=f_2(\gamma_{\mathrm{c}}),\ \sigma_{\mathrm{s}}=f_3(\varepsilon_{\mathrm{s}}). \qquad (5)$$

Substituting geometrical equations (2) and constitutive equations (5) into virtual work equation (4), a corresponding Euler equations would be induced by differential operation as,

$$\begin{cases} \displaystyle\int_{-\frac{H}{2}}^{\frac{H}{2}}\left[B\frac{\mathrm{d}f_1(\varepsilon_{\mathrm{c}})}{\mathrm{d}\varepsilon_{\mathrm{c}}}\frac{\mathrm{d}\varepsilon_{\mathrm{c}}}{\mathrm{d}x}\right]\mathrm{d}z+\sum_{i=1}^{N}\left[A_i\frac{\mathrm{d}f_3(\varepsilon_{si})}{\mathrm{d}\varepsilon_{si}}\frac{\mathrm{d}\varepsilon_{si}}{\mathrm{d}x}\right]=0 \\ \displaystyle\int_{-\frac{H}{2}}^{\frac{H}{2}}\left[Bz\frac{\mathrm{d}f_1(\varepsilon_{\mathrm{c}})}{\mathrm{d}\varepsilon_{\mathrm{c}}}\frac{\mathrm{d}\varepsilon_{\mathrm{c}}}{\mathrm{d}x}\right]\mathrm{d}z+\sum_{i=1}^{N}\left[A_iz_i\frac{\mathrm{d}f_3(\varepsilon_{si})}{\mathrm{d}\varepsilon_{si}}\frac{\mathrm{d}\varepsilon_{si}}{\mathrm{d}x}\right] -BH\kappa f_2(\gamma_{\mathrm{c}})=0 \\ \displaystyle\kappa BH\frac{\mathrm{d}f_2(\gamma_{\mathrm{c}})}{\mathrm{d}\gamma_{\mathrm{c}}}\frac{\mathrm{d}\gamma_{\mathrm{c}}}{\mathrm{d}x}+q(x)=0 \end{cases} \qquad (6)$$

The relative boundary conditions are described as follows,

$$\begin{cases} \left.\left\{\left[\displaystyle\int_{-\frac{H}{2}}^{\frac{H}{2}}Bf_1(\varepsilon_{\mathrm{c}})\mathrm{d}z+\sum_{i=1}^{N}A_if_3(\varepsilon_{si})\right]\delta u_0\right\}\right|_{x_0}^{x_1}=0 \\ \left.\left\{\left[\displaystyle\int_{-\frac{H}{2}}^{\frac{H}{2}}Bf_1(\varepsilon_{\mathrm{c}})z\mathrm{d}z+\sum_{i=1}^{N}A_iz_if_3(\varepsilon_{si})\right]\delta\theta\right\}\right|_{x_0}^{x_1}=0 \\ \left.\left[\kappa BHf_2(\gamma_{\mathrm{c}})\delta w\right]\right|_{x_0}^{x_1}=0 \end{cases} \qquad (7)$$

2.2 Linearizing

If concrete and steel are both linear elasticity materials, some variables can be demonstrated as,

$$\sigma_{\mathrm{c}}(\varepsilon_{\mathrm{c}})=E_{\mathrm{c}}\varepsilon_{\mathrm{c}},\quad \tau_{\mathrm{c}}(\gamma_{\mathrm{c}})=G_{\mathrm{c}}\gamma_{\mathrm{c}},\quad \sigma_{\mathrm{s}}(\varepsilon_{\mathrm{s}})=E_{\mathrm{s}}\varepsilon_{\mathrm{s}} \qquad (8)$$

Above E_c and G_c respectively means elastic modulus and shear modulus of concrete; E_s means elastic modulus of steel.

Then, boundary value problems made by equations (6) and equations (7) are transferred into linear differential system as follows,

$$\begin{cases} E_c A \dfrac{d^2u_0}{dx^2} + \sum_{i=1}^{N}\left[(E_s - E_c)A_i\left(\dfrac{d^2u_0}{dx^2} - z_i\dfrac{d^2\theta}{dx^2}\right)\right] = 0 \\ E_c I\dfrac{d^2\theta}{dx^2} - \sum_{i=1}^{N}\left[(E_s - E_c)A_i z_i\left(\dfrac{d^2u_0}{dx^2} - z_i\dfrac{d^2\theta}{dx^2}\right)\right] \\ \qquad + \kappa G_c A_c\left(\dfrac{dw}{dx} - \theta\right) = 0 \\ \kappa G_c A_c\left(\dfrac{d^2w}{dx^2} - \dfrac{d\theta}{dx}\right) + q(x) = 0 \end{cases} \tag{9}$$

The boundary conditions can be converted into equations as,

$$\left(F_N\delta u_0\right)\Big|_{x_0}^{x_1} = 0,\ \left(F_s\delta w\right)\Big|_{x_0}^{x_1} = 0,\ \left(M\delta\theta\right)\Big|_{x_0}^{x_1} = 0 \tag{10}$$

And, axial force (F_N), shear force (F_s) and bending moment (M) in RC beam are shown as follows,

$$F_N = E_c A_c\frac{du_0}{dx} + (E_s - E_c)\sum_{i=1}^{N}\left[A_i\left(\frac{du_0}{dx} - z_i\frac{d\theta}{dx}\right)\right] \tag{11}$$

$$F_s = \kappa G_c A_c\left(\frac{dw}{dx} - \theta\right) \tag{12}$$

$$M = -E_c I\frac{d\theta}{dx} + (E_s - E_c)\sum_{i=1}^{N}\left[A_i z_i\left(\frac{du_0}{dx} - z_i\frac{d\theta}{dx}\right)\right] \tag{13}$$

If effect of steel is ignored, equations (9) would be degenerated into,

$$\begin{cases} E_c A\dfrac{d^2u_0}{dx^2} = 0 \\ E_c I\dfrac{d^2\theta}{dx^2} + \kappa G_c A_c\left(\dfrac{dw}{dx} - \theta\right) = 0 \\ \kappa G_c A_c\left(\dfrac{d^2w}{dx^2} - \dfrac{d\theta}{dx}\right) + q(x) = 0 \end{cases} \tag{14}$$

A and I mean cross section area and bending moment of inertia; A_c means area of concrete in cross section. It is not difficult to find that equations (14) is statics control functions of classic linear elasticity (Timoshenko beam) [19]. Also, boundary conditions is naturally regressed, while it is not need to be described in this paper.

2.3 First-order ordinary differential equation system

Six new unknown parameters $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$, $y_5(x)$ and $y_6(x)$ are lend in shoot method. Then, Euler equation can be transformed into a first-order ordinary differential equation system $F(y)$ about basic unknown y. The calculating process is list below.

$$\begin{cases} y_1(x) = u_0(x),\ y_2(x) = \dfrac{du_0(x)}{dx},\ y_3(x) = \theta(x), \\ y_4(x) = \dfrac{d\theta(x)}{dx},\ y_5(x) = w(x),\ y_6(x) = \dfrac{dw(x)}{dx} \end{cases} \tag{15}$$

$$F(y)=dy/dx \tag{16}$$

$$y=[\,y_1\quad y_2\quad y_3\quad y_4\quad y_5\quad y_6]^T \tag{17}$$

$$F(y)=[\,F_1(y)\quad F_2(y)\quad F_3(y)\quad F_4(y)\quad F_5(y)\quad F_6(y)]^T \tag{18}$$

$$\begin{cases} F_1(y) = y_2,\ F_2(y) = \dfrac{\alpha_2\alpha_4}{\alpha_2^2 - \alpha_1\alpha_3},\ F_3(y) = y_4, \\ F_4(y) = \dfrac{\alpha_1\alpha_4}{\alpha_2^2 - \alpha_1\alpha_3},\ F_5(y) = y_6,\ F_6(y) = y_4 - \dfrac{q}{\alpha_5} \end{cases} \tag{19}$$

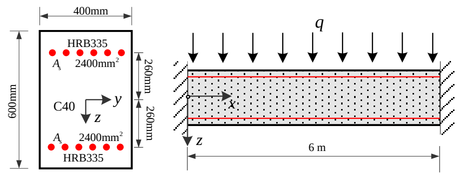
Premeters like α_1, α_2, α_3, α_4 and α_5 are defined by,

$$
\begin{cases}
\alpha_1 = \int_{-H/2}^{H/2}\left[B\dfrac{\mathrm{d}f_1(\varepsilon_\mathrm{c})}{\mathrm{d}\varepsilon_\mathrm{c}}\right]\mathrm{d}z + \sum_{i=1}^{N} A_i \dfrac{\mathrm{d}f_3(\varepsilon_{\mathrm{s}i})}{\mathrm{d}\varepsilon_{\mathrm{s}i}} \\
\alpha_2 = \int_{-H/2}^{H/2}\left[Bz\dfrac{\mathrm{d}f_1(\varepsilon_\mathrm{c})}{\mathrm{d}\varepsilon_\mathrm{c}}\right]\mathrm{d}z + \sum_{i=1}^{N} A_i z_i \dfrac{\mathrm{d}f_3(\varepsilon_{\mathrm{s}i})}{\mathrm{d}\varepsilon_{\mathrm{s}i}} \\
\alpha_3 = \int_{-H/2}^{H/2}\left[Bz^2\dfrac{\mathrm{d}f_1(\varepsilon_\mathrm{c})}{\mathrm{d}\varepsilon_\mathrm{c}}\right]\mathrm{d}z + \sum_{i=1}^{N} A_i z_i^2 \dfrac{\mathrm{d}f_3(\varepsilon_{\mathrm{s}i})}{\mathrm{d}\varepsilon_{\mathrm{s}i}} \\
\alpha_4 = BH\, f_2(\gamma_\mathrm{c}) \\
\alpha_5 = BH\dfrac{\mathrm{d}f_2(\gamma_\mathrm{c})}{\mathrm{d}\gamma_\mathrm{c}}
\end{cases}
\tag{20}
$$

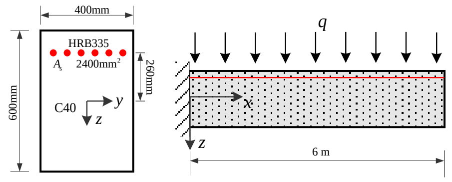
Strain in concrete is demonstrated as $\varepsilon_\mathrm{c}(x, z) = y_2(x) - zy_4(x)$. When it is injected into function (7), boundary conditions are naturally obtained about unknown **y**. Furthermore, integral operation of α_1, α_2 and α_3 can be achieved by Gauss, which results in avoiding dividing steps of fiber needed in fiber cross-section method.

3 Numerical verification and analysis

Static analysis of material nonlinearity in RC beam is recently done based on Euler-Bernoulli kinematic assumption by adopting differential quadrature method [20]. In this paper, results of reference [20] are firstly taken to test validity of nonlinear shooting method program. Secondly, shearing effect is discussed in deep RC beam. Finally, moment modulation coefficients are analyzed under different loads.



(a) clamped-clamped RC beam



(b) clamped-free RC beam

Fig. 2. Boundary conditions and cross-sectional configurations of RC beam

Geometric and physical parameters in two kinds of RC beam are displayed in **Fig. 2**. Cross section area in each beam owns width (B) at 0.4 m, height (H) at 0.6 m and span (L) at 6 m. For clamped-clamped RC beam, HRB335 steels are symmetrically distributed around y-axis in upper and lower limb in cross section. The cross section area of HRB335 steel is at 2400 mm^2, ranging y-axis at 0.26 m. For fixed-free RC beam, structural parameters are same, but it has no lower limb steel inserted. $f_1(\varepsilon_\mathrm{c})$ and $f_3(\varepsilon_\mathrm{s})$ below are assumed about properties of steel and concrete materials in design theory of bridge and culvert [21].

$$
f_1(\varepsilon_\mathrm{c}) = \begin{cases} \sigma_0\left[2\left(\dfrac{\varepsilon_\mathrm{c}}{\varepsilon_0}\right) - \left(\dfrac{\varepsilon_\mathrm{c}}{\varepsilon_0}\right)^2\right], & \varepsilon_0 \le \varepsilon_\mathrm{c} \le 0 \\ \sigma_0, & \varepsilon_\mathrm{cu} \le \varepsilon_\mathrm{c} < \varepsilon_0 \end{cases} \tag{21}
$$

$$
f_3(\varepsilon_\mathrm{s}) = \begin{cases} E_\mathrm{s}\varepsilon_\mathrm{s}, & |\varepsilon_\mathrm{s}| \le \varepsilon_\mathrm{y} \\ \sigma_y, & |\varepsilon_\mathrm{s}| > \varepsilon_\mathrm{y} \end{cases} \tag{22}
$$

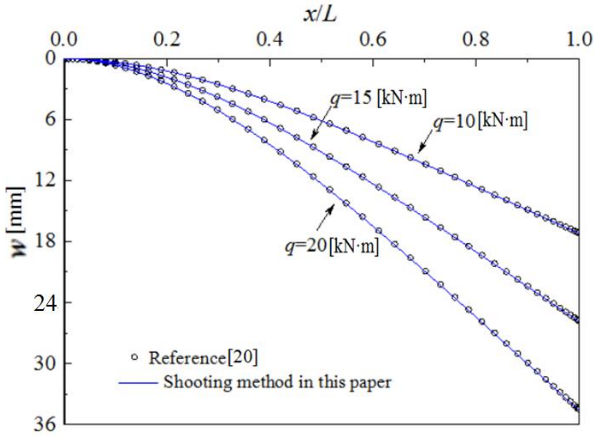
But reference [21] does not take shear deformation of RC beam into account. Therefore, shearing stress of concrete is assumed by a linear relationship with shear deformation as,

$$f_2(\gamma_\mathrm{c}) = G_\mathrm{c}\gamma_\mathrm{c} \tag{23}$$

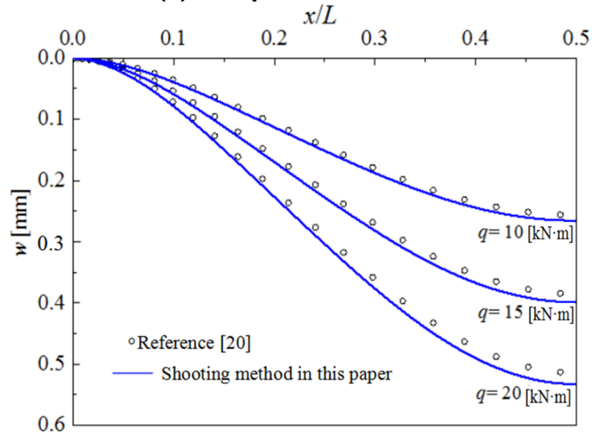
In Fig.2, C40 concrete is shown with σ_0 =-22.78MPa, ε_0 = -0.002, ε_cu = -0.0035 and G_c = 20.5GPa, with resistance to tension not calculated. HRB335 steel has yield stress σ_y = 335MPa and elastic modulus E_s = 200GPa.

3.1 Numerical verification

The curves in Fig.3 describe deflection (w) analysis of RC beam under loads at 10kN·m, 15kN·m and 20 kN·m, respectively. They are all calculated based on kinematics assumption of Euler-Bernoulli and Timoshenko assumptions. Because of a bigger effective ratio of length to height (λ=10) in clamped-free RC beam, results in Fig.3a approach to results in reference [20]. Results shown in Fig.3b display a certain shear effect with a smaller value of λ at 5. It therefore causes a little difference of deflection analysis. It can be concluded that results in Fig.3 to some extent verify correctness of established functions and numerical program.



(a) clamped-free RC beam



(b) clamped-clamped RC beam

Fig. 3. RC beam deflections based on Euler-Bernoulli and Timoshenko assumptions

3.2 Shear effect

Effect of shearing deformation on structural mechanics behavior should not be neglected in deep beam. So, developed shooting method program is used to study shear effect in this paper.

Taken characteristics of nonlinear material into consideration, stiffness of RC beam will change at different loads. Additionally, degree of depth in beam also needs to be considered. For considering shear effect of beam component and synthesizing above two factors, parameter n is induced as,

$$n=q(L/H)^2 \qquad (24)$$

Table 1. Mid-span deflection of fixed-fixed RC beam (mm, n=1500kN·m)

L/H	q [kN·m]	Timoshenko [mm] (a)	Euler-Bernoulli [mm] (b)	error [%] [(a)-(b)]/(a)
25	2.4	2.419	2.395	1.01
20	3.75	1.554	1.533	1.35
15	6.67	0.881	0.863	2.05
10	15	0.399	0.383	3.94
5	60	0.110	0.096	12.93

In clamped-clamped RC beam, analyzed errors by shear effect are found in Table 1. The values (a) are calculated by shooting method, while those (b) are similarly calculated with increased shear modulus of concrete to 10^5G_c. It can be seen that error made by shearing deformation goes up by decreased value of L/H. When value of L/H is 5, shearing error arrives at 12.93%. For required precision of civil engineering analysis, Euler-Bernoulli model is carefully applied when ratio of length to height of beam is below 10 in this calculation.

3.3 Moment modulation coefficients

To simplify design and analysis for RC beam structure, engineers usually use linear models to solve internal force problems. The linear solutions are used to be multiplied by amplitude modulation coefficient of inner force. Then, the gotten results are used to be approximate to designed inner force of nonlinear components. In this text, linear and nonlinear moment distribution are respectively solved to acquired corresponding moment modulation coefficients.

Taking fixed-fixed RC beam for example, curve between load and mid-displacement of structure in Fig.2a is demonstrated in Fig.4. The topside load of this curve means ultimate load (q_u) at 357.0kN. For investigating M distribution at each stage of deformation process as soon as possible, three loads are adopted at 0.1q_u, 0.7q_u and 0.9q_u to solve relative moment distribution. These stages include approximate linear stage, yielding phase, and approximate limit period. bending moment conditions are shown in Fig.5 under three loads. It can be seen from **Fig.6** that effect of material nonlinearity does not obviously perform at 0.1q_u, while moment modulation coefficient is 0.96 in mid-section. When loads reach at 0.7q_u and 0.9q_u, moment modulation coefficients are 0.87 and 0.73. These obtained moment modulation coefficients are similar to those suggested values in Design Code for Concrete Structure in China [22]. Above analysis means established way in this paper is reliable. As well, it supplies a new reference way to design RC beam structure.

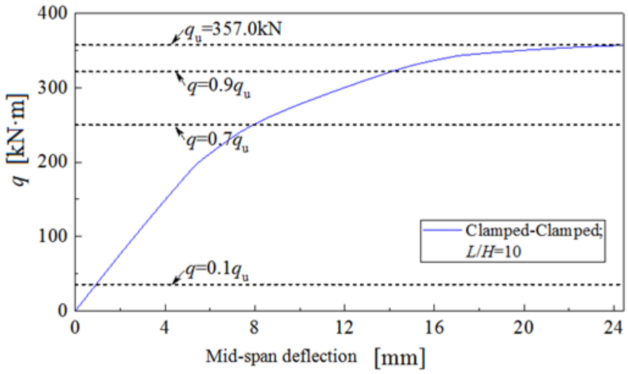
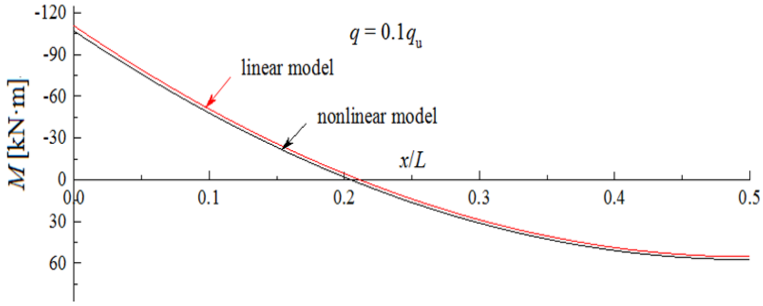
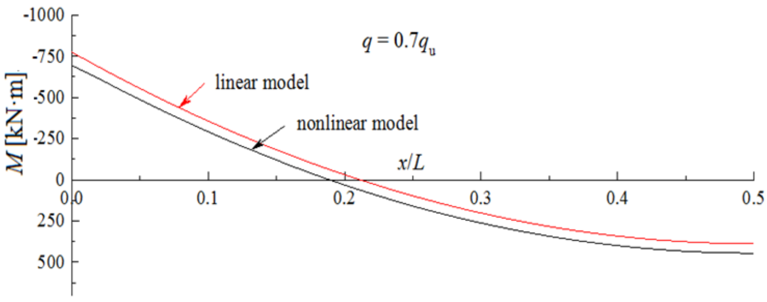


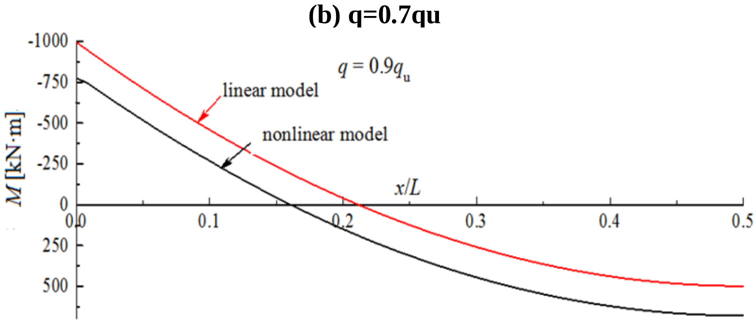
Fig. 4. Load-displacement curve of fixed-fixed RC beam



(a) q=0.1qu



(b) q=0.7qu



(c) q=0.9qu

Fig.5. Distribution of linear and nonlinear bending moment in fixed-fixed RC beam under different loads

4 Conclusion

By applying virtual work principle of static response in RC beam, there are control differential equations and edge conditions set up with material nonlinearity and first order shear deformation taken into account. The linearizing control function is in accordance with classic equation, indirectly testing accuracy of induced equations in this paper. The obtained high-order differential equations are transformed into a standard first-order ordinary differential equation system, subsequently solved by nonlinear shooting method. With verification and analysis done by numerical program of shooting method, it can be firstly concluded that comparison of deflection analysis to Euler-Bernoulli RC beam model verifies established Timoshenko model and corresponding shooting method program. Secondly, analysis of shearing effect reveals that it is necessary to have continuous consideration about shearing deflection based on Euler-Bernoulli for deeper RC beam. Thirdly, amplitude modulation coefficients got by numerical example of moment modulation has a nicer coincidence degree with suggested values in concrete design criterion. It continuously indicates reasonableness of gotten procedure in this paper. Finally, acquired simple shooting method in this paper will be a supplement way to analyze finite element.

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